Reduced-Order Models in FDTD

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***Abstract*—This letter describes a new hybrid technique for enhancing the efficiency of the finite difference time domain method (FDTD) by incorporation with in Yee's mesh a macromodel generated by an order reduction algorithm applied to the finite difference frequency domain (FDFD) wave equation defined on a higher resolution grid. This approach increases the accuracy of the calculations and reduces the overall simulation time.**

***Index Terms*—Finite-difference time-domain method, reduced-order model.**

I. Introduction

In THE last few years, reduced-order modeling has become the method of choice for finding compact representations of large passive electronic circuits such as interconnects [1]. The technique involves finding a low-order Padé approximant to the original transfer function in a sense that it matches, up to a certain degree, the leading expansion coefficients (so-called moments) of the Taylor series representation of the approximated function. In computational electromagnetics, the order-reduction techniques have been proposed [2] for fast frequency sweep in the computationally expensive techniques such as the finite-element method. The time consuming decomposition of the large sparse matrix has to be performed only at one or few frequency points, and this decomposition is used to create the reduced-order model, which then allows one to reconstruct the circuit characteristics at other frequencies. The reduced-order models have also been derived from FDTD equations [3]. Initially, the reduced-order models were constructed by considering Maxwell's equations in the whole computational domain so, in fact, the circuit characteristics were derived from the model. Recently, Zhu and Cangellaris [4] applied reduced-order modeling instead of local grid refinement in the finite element method. The generalized impedance matrix representation of a subvolume containing small feature was derived and used to create macro-elements, which were then embedded into the finite-element mesh. In this letter, we show how reduced-order models derived from frequency domain Maxwell's equations can be used to represent subvolumes in the finite-difference time do main method.

II. Model-Order Reduction Technique

Several techniques have been proposed to derive the reduced-order models. To ensure the stable operation of the FDTD with the macromodel embedded in the subvolume, we used the efficient nodal order reduction (ENOR) approach, which leads to passive and reciprocal representation of the transfer function of a multiport. This technique is suitable for the analysis of circuits whose Laplace transformed N nodal equation for p input and p output ports are given by

$$\left(s\underline{\underline{C}} + \underline{\underline{G}} + \underline{\underline{\Gamma}}/s\right)\underline{X}(s) = \underline{\underline{B}}\,\underline{J}(s) \tag{1}$$

where

$\underline{\underline{C}}, \underline{\underline{G}}, \underline{\underline{\Gamma}} \in R^{N\times N}$	symmetric semi-positive matrices describing circuit reactive or resistive components,
$\underline{X} \in C^N$	vector of nodal voltages,
$\underline{J} \in C^p$	vector of current sources at p ports whose location is described by the incidence matrix $\underline{\underline{B}} \in R^{N\times p}$.

For lossless systems, the $p\times p$ impedance matrix-valued transfer function is

$$\underline{\underline{Z}}(s) = \underline{\underline{B}}^T\left(s\underline{\underline{C}} + \underline{\underline{\Gamma}}/s\right)^{-1}\underline{\underline{B}}. \tag{2}$$

In the ENOR technique, the mth matrix-Padé approximant to $\underline{\underline{Z}}(s)$ is expressed by

$$\underline{\underline{Z}}_m(s) = \underline{\underline{B}}_m{}^T\left(s\underline{\underline{C}}_m + \underline{\underline{\Gamma}}_m/s\right)^{-1}\underline{\underline{B}}_m \tag{3}$$

where lower-order matrices appearing in the above equations are obtained by orthogonal projection with respect to the basis $\underline{\underline{V}} \in R^{N\times m}$

$$\underline{\underline{C}}_m = \underline{\underline{V}}^T\underline{\underline{C}}\,\underline{\underline{V}} \quad \underline{\underline{B}}_m = \underline{\underline{V}}^T\underline{\underline{B}}, \quad \underline{\underline{\Gamma}}_m = \underline{\underline{V}}^T\underline{\underline{\Gamma}}\,\underline{\underline{V}}. \tag{4}$$

The basis $\underline{\underline{V}}$ is obtained by successively matching block moments of the nodal voltages expanded about the selected frequency s_0 (see [5] for details of the computational procedure). The advantage of ENOR lies in the fact that the model generated is reciprocal and passive.

III. Creating Reduced- Order Models for FDTD

The reduced-order models are created in the frequency domain, while the FDTD is a marching-in-time algorithm. Moreover, the FDTD technique operates on a staggered grid and the macromodel has to account for that. For these reasons, the finite-difference frequency-domain (FDFD) Maxwell's grid equations with electric and magnetic fields shifted half a discretization step with respect one to another were chosen t o generate the model. This field arrangement is compatible with the FDTD mesh. The Laplace transforms of Maxwell's grid equations are then

$$\underline{\underline{R}}_e\underline{E} = s\underline{\underline{D}}_\mu\underline{H} \tag{5}$$
$$-\underline{\underline{R}}_h\underline{H} = s\underline{\underline{D}}_\epsilon\underline{E} \tag{6}$$

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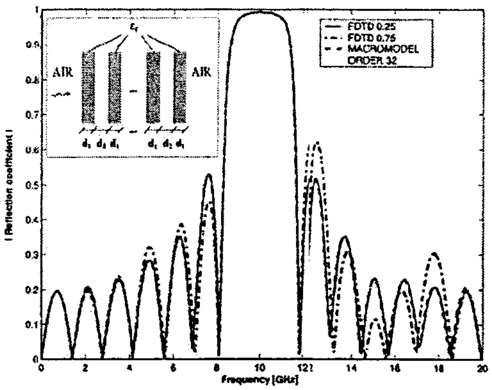

Fig. 1. Magnitude of the plane wave reflection coefficient for a geometry shown in the inset. See the text for explanations.

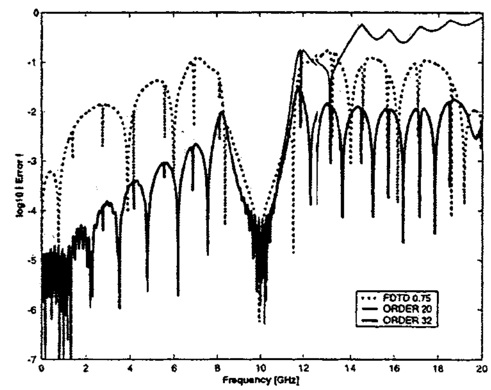

Fig. 2. Absolute error of the simulations (in a logarithmic scale) relative to the fine grid FDTD calculations with $dz = 0.25$ mm. See the text for explanations.

where $\underline{\underline{R}}_e = \underline{\underline{R}}_h^T$ are the matrices corresponding to the curl operators in the continuous space and $\underline{\underline{D}}_\mu$, $\underline{\underline{D}}_\epsilon$ are diagonal matrices describing the permeability and permittivity distribution inside the computational volume. The above equations can easily be converted to the symmetric form required by the ENOR algorithm. Introducing the normalization $\underline{e} = \sqrt{\underline{\underline{D}}_\epsilon}\underline{E}$ and $\underline{h} = \sqrt{\underline{\underline{D}}_\mu}\underline{H}$ one gets from Maxwell's grid equations

$$\left(s\underline{\underline{I}} + \underline{\underline{R}}_e^T \underline{\underline{R}}_e / s\right) \underline{e} = 0. \tag{7}$$

It is seen that by adding suitable source terms in appropriate grid points determined by the incidence matrix $\underline{\underline{B}}$ in the form of magnetic field, the above equation becomes an analog of (1) with $\underline{\underline{C}} = \underline{\underline{I}}$ the identity matrix, and $\underline{\underline{\Gamma}} = \underline{\underline{R}}_e^T \underline{\underline{R}}_e$. The source terms are the magnetic fields which will be provided by FDTD when the model is embedded within the time marching scheme.

Once the orthogonal basis $\underline{\underline{V}}$ is constructed in the manner described in [5], the reduced-order model is converted into the residue/pole representation, which is suitable for deriving the time marching equations for the model. To this end, the eigendecomposition of the reduced matrix $\underline{\underline{\Gamma}}_m$ is carried out as follows

$$\underline{\underline{\Gamma}}_m = \underline{\underline{S}}\,\underline{\underline{\Omega}}^2 \underline{\underline{S}}^T \tag{8}$$

where $\underline{\underline{S}}$ and $\underline{\underline{\Omega}}^2$ are m by m matrices consisting of respectively eigenvectors and eigenvalues of matrix $\underline{\underline{\Gamma}}_m = \underline{\underline{V}}^T \underline{\underline{R}}_e^T \underline{\underline{R}}_e \underline{\underline{V}}$. Matrix $\underline{\underline{\Omega}}^2$ is a diagonal one. This allows us to represent reduced-order matrix-valued impedance transfer function as

$$\underline{\underline{Z}}_m(s) = \sum_{k=1}^{m} \frac{s\underline{\underline{z}}_k}{s^2 + \omega_k^2} \tag{9}$$

where $\underline{\underline{z}}_k$ is a p by p matrix whose entries are obtained by multiplying the kth column of $\underline{\underline{B}}_m^T \underline{\underline{S}}$ with the kth row of $\underline{\underline{S}}^T \underline{\underline{B}}_m$.

The residue/pole representation of the matrix-valued impedance transfer function can readily be used to derive the set of differential equation in time domain which describe the relation between the electric and magnetic fields at the ports. Discretizing these differential equations and bearing in mind that the FDTD requires the E and H fields to be computed at the instances that are half a time step off, one gets the following discretized version of the time domain evolution of the signals at the reduced-order model ports (denoted by $\underline{\mathcal{E}}^{n+1}$ and $\underline{\mathcal{H}}^{n+1/2}$)

$$\underline{\mathcal{E}}^{n+1} = \sum_{k=1}^{m} \Re e\, \underline{u}_k^{n+1} \tag{10}$$

where

$$\underline{u}_k^{n+1} = \underline{\underline{A}}_k \underline{\mathcal{H}}^{n+1/2} + \underline{\underline{B}}_k \underline{u}_k^n \tag{11}$$

and the matrix-valued coefficients defined as

$$\underline{\underline{A}}_k = \frac{2\Delta t}{2 + j\Delta t \omega_k} \underline{\underline{z}}_k \quad \underline{\underline{B}}_k = \frac{2 - j\Delta t \omega_k}{2 + j\Delta t \omega_k} \tag{12}$$

with Δt denoting time step. If the macromodel is generated from the grid equations in frequency domain in such a way that the ports are defined at the locations of appropriate normalized neighboring E and H fields of the FDTD grids, then the equations (10) and (11) are used at the boundary between FDTD and the subvolumes represented by the model.

IV. Results

As an example, we shall apply the above technique to find the reflection coefficient of a plane wave impinging from air on a structure consisting of 7 identical 5.056 mm thick layers of dielectric of $\varepsilon_r = 2.2$, each separated from another by a 7.5 mm stratum of air. The structure was analyzed three times in time domain and the magnitudes of the reflection coefficient s obtained are shown in Fig. 1. First the fine grid ($dz = 0.25$ mm) FDTD calculations were carried out and these results were taken as reference (solid line in Fig. 1). The same FDTD algorithm was rerun with coarse grid $dz = 0.75$ mm (dashed-dotted line). It is apparent that the grid was too coarse to produce the correct result. Finally, the reduced-order model of the reflecting part was created with FDFD and $dz = 0.25$ mm and $s_0 = 8\pi 10^9$. The model was then embedded into the coarse FDTD grid $dz = 0.75$ mm, which covered only the free-space. Results from the coarse FDTD with the embedded ENOR model of order $m = 32$ are indistinguishable from the fine grid simulations. For clarity, the absolute errors with reference to the fine grid calculations are shown in Fig. 2 in a logarithmic scale. The curves correspond to the coarse grid FDTD (dotted line) and coarse grid FDTD

TABLE I
NORMALIZED ERROR NORMS FOR DIFFERENT MODEL ORDERS

Method used		Normalized error norm w.r.t.	
FDTD dz	model	FDTD dz=025	Analytical
0.25mm	no	0	0.033
0.75mm	no	0.137	0.150
0.75mm	m= 20	0.624	0.619
0.75mm	m= 28	0.133	0.134
0.75mm	m= 32	0.016	0.043
0.75mm	m= 40	0.016	0.044

combined with fine grid ENOR models for $m = 20$ (thin solid line) and $m = 32$ (thick solid line). It is seen that increasing the model order increases its bandwidth. Table I shows the L_2 norms for the absolute error w.r.t. fine grid FDTD calculations (third column) and the analytical solution (fourth column) normalized to the solution norm.

Finally, the speedup factor depends on the resolution ratio between the coarse and fine grid used to create the model and the volume replaced by the macromodel. In our example, the speedup of the simulations using model order $m = 32$ relative to fine grid calculation is about 6 (95 versus 16 seconds). A word of comment is needed regarding the maximal allowable time step when reduced-order models are embedded into the FDTD mesh. Its maximal value depends on the location of pole ω_m, but is always larger than the time step in the fine grid simulations. It is often possible to construct the model in such a way that coarse grid FDTD with the fine grid macromodel embedded operates with the CFL time step of the coarse grid.

V. CONCLUSION

We have demonstrated the application of the model order reduction technique in the FDTD calculations. The reduced-order model constructed using the fine grid FDFD formulation can be embedded into the coarse FDTD grid. The accuracy of the FDTD calculations using the ENOR model to represent a complex subvolumes was found to be very good. The technique opens up new possibilities to increase the computational efficiency of the FDTD.

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